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SHARP AND PRINCIPAL ELEMENTS IN EFFECT ALGEBRAS

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Abstract. In this paper we characterize the effect algebras whose sharp and principal elements coincide. We present partially solution to the problems: when the set of sharp (principal) elements is closed under orthosum. We also give examples of two non-isomorphic effect algebras having the same universum, partial order and orthosupplementation.

1. **Introduction**

Effect algebras have been introduced by Foulis and Bennet in 1994 (see [5]) for the study of foundations of quantum mechanics (see [4]). Independtly, Chovanec and Kôpka introduced an essentially equivalent structure called *D-poset* (see [9]). Another equivalent structure was introduced by Giuntini and Greuling in [6]).

The most important example of an effect algebra is $(E(H), 0, I, \oplus)$, where H is a Hilbert space and $E(H)$ consists of all self-adjoint operators A on H such that $0 \leq A \leq I$. For $A, B \in E(H)$, $A \oplus B$ is defined if and only if $A+B \leq I$ and then $A \oplus B = A+B$. Elements of $E(H)$ are called *effects* and they play an important role in the theory of quantum measurements ([2],[3]).

A quantum effect may be treated as two-valued (it means 0 or 1) quantum measurement that may be unsharp (fuzzy). If there exist some pairs of effects a, b which posses an orthosum $a \oplus b$ then this orthosum correspond to a parallel measurement of two effects.

In this paper we solved the following Open Problem: Characterize the effect algebras whose sharp and principal elements coincide (see [8]). So far it was known (see Theorem 3.16 in [1]) that if effect algebra E is lattice-ordered then $e \in E$ is principal iff $e \wedge e' = 0$. It also was known that in every effect algebra any principal element is sharp (see Lemma 3.3 in [7]).

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Definition 1.1. In [5] an *effect algebra* is defined to be an algebraic system $(E, 0, 1, \oplus)$ consisting of a set E, two special elements $0, 1 \in E$ called the *zero* and the *unit*, and a partially defined binary operation $\oplus$ on E that satisfies the following conditions for all $p, q, r \in E$:

(1) [Commutative Law] If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.
(2) [Associative Law] If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ is defined, $(p \oplus q) \oplus r$ is defined, and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.
(3) [Orthosupplementation Law] For every $p \in E$ there exists a unique $q \in E$ such that $p \oplus q$ is defined and $p \oplus q = 1$.
(4) [Zero-unit Law] If $1 \oplus p$ is defined, then $p = 0$.

For simplicity , we often refer to E, rather than to $(E, 0, 1, \oplus)$, as being an effect algebra.

If $p, q \in E$, we say that p and q are orthogonal and write $p \perp q$ iff $p \oplus q$ is defined in E. If $p, q \in E$ and $p \oplus q = 1$, we call q the *orthosupplement* of p and write $p' = q$.

It is shown in [5] that the relation $\leq$ defined for $p, q \in E$ by $p \leq q$ iff $\exists r \in E$ with $p \oplus r = q$ is a partial order on E and $0 \leq p \leq 1$ holds for all $p \in E$. It is also shown that the mapping $p \mapsto p'$ is an order-reversing involution and that $q \perp p$ iff $q \leq p'$. Furtheremore, E satisfies the following *cancellation law*: If $p \oplus q \leq r \oplus q$, then $p \leq r$.

An element $a \in E$ is *sharp* if the greatest lower bound of the set $\{a, a'\}$ equals 0 (i.e. $a \wedge a' = 0$). We denote the set of sharp elements of E by S_E.

An element $a \in E$ is said to be *principal* iff for all $p, q \in E$, $p \perp q$ and $p, q \leq a \Rightarrow p \oplus q \leq a$. We denote the set of principal elements of E by P_E.

Definition 1.2. For effect algebras E_1, E_2 a mapping $\phi \colon E_1 \to E_2$ is said to be an *isomorphism* if ϕ is a bijection, $a \perp b \iff \phi(a) \perp \phi(b)$, $\phi(1) = 1$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$.

Let us observe that if $\phi \colon E_1 \to E_2$ is an isomorphism then $\phi(0) = 0$, because $\phi(0) \oplus 0 = \phi(0) = \phi(0 \oplus 0) = \phi(0) \oplus \phi(0)$ so by cancellation law $0 = \phi(0)$.

Definition 1.3. A quasigroup $(Q, \cdot)$ consists of a non-empty set Q equipped with a one binary operation $\cdot$ such that if any two of a, b, c are given elements of a quasigroup, $ab = c$ determines the third uniquely as an element of the quasigroup.

Moreover if $a \cdot b = c \iff c \cdot a = b$ then Q is called *semisymmetric* (see [10]). Commutative semisymmetric quasigroups are called *totally symmetric* (see [11]).

2. Main Theorem

In order to characterize principal elements we need the following known Theorem:

Theorem 2.1. [7, Theorem 3.5] *If $p, q \in E$, $p \perp q$, and $p \vee q$ exists in E, then $p \wedge q$ exists in E, $p \wedge q \leq (p \vee q)' \leq (p \wedge q)'$ and $p \oplus q = (p \wedge q) \oplus (p \vee q)$.*

Lemma 2.2. *Let $(E, 0, 1, \oplus)$ be an effect algebra. If $x \in P_E$, $t \in E$ and $t \leq x$ then there exists $t \vee x'$ in E and*

$$t \vee x' = t \oplus x'.$$

Proof. Suppose that $x \in P_E$. Let $t \in E$ and $t \leq x$ hence $t \perp x'$. We show that $t \oplus x'$ is the join of t and x'.

Obviously $t \leq t \oplus x'$ and $x' \leq t \oplus x'$. Suppose that $u \in E$, $t \leq u$ and $x' \leq u$ then

$$t \perp u' \tag{1}$$

and

$$u' \leq x \quad t \leq x. \tag{2}$$

Now (1) and (2) implies $t \oplus u' \leq x$ since $x \in P_E$. Hence $x' \perp (t \oplus u')$ and by associativity $x' \perp t$ and $(x' \oplus t) \perp u'$ thus $t \oplus x' \leq u$ so $t \oplus x'$ is the smallest upper bound of the set $\{t, x'\}$ thus $t \oplus x' = t \vee x'$. □

It turns out that under some conditions every effect algebra satisfies the de Morgan's law.

Lemma 2.3. *Let $(E, 0, 1, \oplus)$ be an effect algebra. If $x, y \in E$ and there exists $x \vee y$ in E then there exists $x' \wedge y'$ in E and*

$$x' \wedge y' = (x \vee y)'$$

Proof. Let $x, y \in E$ and suppose that there exists $x \vee y$ in E.

We show that

$$x' \wedge y' = (x \vee y)' \tag{3}$$

We show that $(x \vee y)'$ is a lower bound of the set $\{x', y'\}$: $x \leq x \vee y \Rightarrow x' \geq (x \vee y)'$ and $y \leq x \vee y \Rightarrow y' \geq (x \vee y)'$.

If v is a lower bound of $\{x', y'\}$ then $x' \geq v$, $y' \geq v$ thus $x \leq v'$ and $y \leq v'$ hence $x \vee y \leq v'$ and $(x \vee y)' \geq v$ and it implies that $(x \vee y)'$ is the greatest lower bound of the set $\{x', y'\}$ so (3) is satisfied.

□

Similarly we obtain

Lemma 2.4. *Let $(E, 0, 1, \oplus)$ be an effect algebra. If $x, y \in E$ and there exists $x \wedge y$ in E then there exists $x' \vee y'$ in E and*

$$x' \vee y' = (x \wedge y)'$$

Theorem 2.5. *Let $(E, 0, 1, \oplus)$ be an effect algebra. Then*

$$P_E = \{x \in E \colon x \in S_E \text{ and } \forall_{t \in E} t \leq x \Rightarrow t \vee x' \text{ exists in } E\}$$

Proof. Suppose that $x \in P_E$ then $x \in S_E$ (see Lemma 3.3 in [7]). Let $t \in E$ and $t \leq x$. Then there exists $t \vee x'$ in E by Lemma 2.2.

Suppose that $x \in S_E$ and

$$\forall_{t \in E} t \leq x \Rightarrow t \vee x' \text{ exists in } E. \tag{4}$$

We show that $x \in P_E$.

If $u, s \in E$, $u \leq x$, $s \leq x$ and $u \perp s$ then

$$u \wedge x' = 0 \tag{5}$$

because: if $y \leq x'$ and $y \leq u \leq x$ then $y = 0$ since $x \wedge x' = 0$.

Moreover $u \leq x$ so $u \vee x'$ exists by (4). By Theorem 2.1

$$u \oplus x' = (u \wedge x') \oplus (u \vee x') \overset{(5)}{=} u \vee x' \tag{6}$$

By Lemma 2.3 we have

$$u' \wedge x = (u \vee x')'. \tag{7}$$

Moreover $s \leq u'$ (since $u \perp s$) and $s \leq x$ so $s \leq u' \wedge x$. Hence by (6) and (7) we have

$$s \leq u' \wedge x = (u \vee x')' = (u \oplus x')'$$

so $s \perp (u \oplus x')$ and by associativity $s \oplus u \perp x'$ hence $s \oplus u \leq x$ and $x \in P_E$.

□

In the following theorem we prove that in every effect algebra E sharp and principal elements coincide if and only if there exists in E join of every two orthogonal elements such that one of them is sharp.

Theorem 2.6. *Let $(E, 0, 1, \oplus)$ be an effect algebra. Then $S_E = P_E$ if and only if*

$$\forall_{t,x \in E} (\, t \perp x' \text{ and } x \wedge x' = 0) \Rightarrow t \vee x' \text{ exists in } E \tag{8}$$

Proof. Suppose that $S_E = P_E$. We show that (8) is satisfied.

Let $x, t \in E$, $t \perp x'$ and $x \wedge x' = 0$. Then $t \leq x$, $x \in P_E$ and by Theorem 2.5 we know that $t \vee x'$ exists in E.

Suppose that condition (8) is fullfilled. Obviously $P_E \subseteq S_E$ (see Lemma 3.3 in [7]).

Now our task is to show that $S_E \subseteq P_E$. Let $x \in S_E$. If $t \in E$, $t \leq x$ then $t \perp x'$ and by condition (8) $t \vee x'$ exists in E hence $x \in P_E$ by Theorem 2.5. Thus $S_E \subseteq P_E$. □

Lemma 2.7. *Let $(E, 0, 1, \oplus)$ be an effect algebra. If P_E is closed under $\oplus$ (that is, if $x, y \in P_E$ and $x \perp y$, then $x \oplus y \in P_E$) then*

$$\forall_{x,y \in P_E} \quad x \perp y \Rightarrow x' \wedge (x \oplus y) = y.$$

Proof. Let $x, y \in P_E$ and $x \perp y$. Then $y \leq x'$ and $y \leq x \oplus y$ so y is a lower bound of x' and $x \oplus y$. Let t be a lower bound of x' and $x \oplus y$. We show that $t \leq y$. We know that $t \leq x'$ so $t \perp x$. Moreover $t \leq x \oplus y$ and $x \leq x \oplus y$ hence

$$x \oplus t \leq x \oplus y$$

since $x \oplus y \in P_E$. After using cancellation law we obtain $t \leq y$. Hence y is the largest lower bound of x' and $x \oplus y$ so $x' \wedge (x \oplus y) = y$. □

Lemma 2.8. *Let $(E, 0, 1, \oplus)$ be an effect algebra. If for every $x, y \in S_E$ such that $x \perp y$ there exists $x \vee y$ in E and*

$$x' \wedge (x \vee y) = y, \quad x \oplus y = x \vee y \tag{9}$$

then S_E is closed under $\oplus$ (that is, if $x, y \in S_E$ and $x \perp y$, then $x \oplus y \in S_E$).

Proof. Let $x, y \in S_E$ and $x \perp y$. By (9) we have

$$\begin{aligned} 0 \quad &= y' \wedge y = y' \wedge (x' \wedge (x \vee y)) \overset{Lemma\ 2.3}{=} (x \vee y)' \wedge (x \vee y) \\ &= (x \oplus y)' \wedge (x \oplus y), \end{aligned}$$

so $x \oplus y \in S_E$ and S_E is closed under $\oplus$. □

In the following Theorem we show that if $S_E = P_E$ then $S_E = P_E$ is closed under $\oplus$ if and only if elements in $S_E = P_E$ satisfy the orthomodular law. It partially solves Open problems 3.2 and 3.3 in [8].

Theorem 2.9. *Let $(E, 0, 1, \oplus)$ be an effect algebra such that $S_E = P_E$. Then $S_E = P_E$ is closed under $\oplus$ if and only if for every $x, y \in S_E$ we have*

$$x \leq y \Rightarrow x \vee (x' \wedge y) = y.$$

Proof. Suppose that $S_E = P_E$ is closed under $\oplus$. Let $x, y \in S_E$ and $x \leq y$ then $x \perp y'$ and by Lemma 2.7 we have $x' \wedge (x \oplus y') = y'$ since

$y' \in S_E$. It follows that

$$y \overset{Lemma\ 2.4}{=} x \vee (x \oplus y')' \overset{Lemma\ 2.2}{=} x \vee (x \vee y')'$$

$$\overset{Lemma\ 2.4}{=} x \vee (x' \wedge y).$$

Suppose that for every $x, y \in S_E$ we have

$$x \leq y \Rightarrow x \vee (x' \wedge y) = y \qquad (10).$$

We show that for every $x, y \in S_E$ such that $x \perp y$ there exists $x \vee y$ in E and

$$x' \wedge (x \vee y) = y, \quad x \oplus y = x \vee y.$$

Let $x, y \in S_E$ and $x \perp y$. Then $x \leq y'$ and $y' \in S_E = P_E$, so $x \vee (y')' = x \vee y$ exists in E and $x \vee y = x \oplus y$ by Lemma 2.2. Moreover $x \vee (x' \wedge y') = y'$ by (10). Hence $x' \wedge (x' \wedge y')' = y$ by Lemma 2.3 and $x' \wedge (x \vee y) = y$ by Lemma 2.4. Therefore S_E is closed under $\oplus$ by Lemma 2.8.

□

Let us observe that by Theorem 2.5 principal elements in an effect algebra are determined by partial order $\leq$ and orthosupplementation $'$. We will see that there exist effect algebras $E_1 = (E, 0, 1, \oplus_1)$ and $E_2 = (E, 0, 1, \oplus_2)$ such that orthosupplementation $'$ in E_1 and orthosupplementation $'$ in E_2 are equal and also the same is true for partial order $\leq$, but E_1 and E_2 are not isomorphic.

Definition 2.10. Let $(Q, \cdot)$ be a totally symmetric quasigroup.

We define $E(Q, \cdot) := \Big((Q \times \{0\}) \cup (Q \times \{1\}) \cup \{0\} \cup \{1\}, 0, 1, \oplus\Big)$ where

- $(q_1, 0) \oplus (q_2, 0) = (q_1 \cdot q_2, 1)$ for all $q_1, q_2 \in Q$,
- $(q, 0) \oplus (q, 1) = (q, 1) \oplus (q, 0) = 1$ for all $q \in Q$,
- $0 \oplus x = x \oplus 0 = x$ for all $x \in (Q \times \{0\}) \cup (Q \times \{1\}) \cup \{0\} \cup \{1\}$.

In the remaining cases orthosum $x \oplus y$ is not defined.

Theorem 2.11. *If $(Q, \cdot)$ is a totally symmetric quasigroup then $E(Q, \cdot)$ is an effect algebra.*

Proof. The Commutative Law and Zero-unit Law are obvious. If $q \in Q$ then there exists a unique element $x = (q, 1)$ such that $(q, 0) \oplus x = 1$ so $(q, 0)' = (q, 1)$. Similarly $(q, 1)' = (q, 0)$ so the Orthosupplementation Law is satisfied.

It remains to show that the Associative Law is also fullfilled. Let $x, y, z \in (Q \times \{0\}) \cup (Q \times \{1\}) \cup \{0\} \cup \{1\}$. If $x = 0$ or $y = 0$, or $z = 0$

then The Asociative Law is true. If $y \oplus z$ is defined and $x \oplus (y \oplus z)$ is defined and $x, y, z \neq 0$ then $x, y, z \in Q \times \{0\}$, so there exist $p, q, r \in Q$ such that $x = (p, 0)$, $y = (q, 0)$, $z = (r, 0)$, so $(q, 0) \oplus (r, 0)$ is defined and $(p, 0) \oplus ((q, 0) \oplus (r, 0))$ is defined, then $(p, 0) \oplus (q \cdot r, 1)$ is defined so $q \cdot r = p$ hence $p \cdot q = r$ thus $(p \cdot q, 1) \oplus (r, 0)$ is defined so $((p, 0) \oplus (q, 0)) \oplus (r, 0)$ is defined and $(p, 0) \oplus ((q, 0) \oplus (r, 0)) = ((p, 0) \oplus (q, 0)) \oplus (r, 0) = 1$. Therefore $(x \oplus y) \oplus z$ is defined and $x \oplus (y \oplus z) = (x \oplus y) \oplus z = 1$.

$\square$

Example 2.12. Let $Q = \{1, 2, 3\}$ and

$\cdot_1$	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

$\cdot_2$	1	2	3
1	2	1	3
2	1	3	2
3	3	2	1

then $E(Q, \cdot_1)$ and $(Q, \cdot_2)$ are totally symmetric quasigroups (see Example 2 and 3 in [11]). Then by Theorem 2.11 $E(Q, \cdot_1)$ and $E(Q, \cdot_2)$ are effect algebras with the following $\oplus$ tables. In this tables we do not include 0 and 1, since they have trivial sums and a dash means that the corresponding $\oplus$ is not defined:

$\oplus_1$	a_1	a_2	a_3	a'_1	a'_2	a'_3
a_1	a_1	a_3	a_2	1	−	−
a_2	a_3	a_2	a_1	−	1	−
a_3	a_2	a_1	a_3	−	−	1
a'_1	1	−	−	−	−	−
a'_2	−	1	−	−	−	−
a'_3	−	−	1	−	−	−

$\oplus_2$	a_1	a_2	a_3	a'_1	a'_2	a'_3
a_1	a_2	a_1	a_3	1	−	−
a_2	a_1	a_3	a_2	−	1	−
a_3	a_3	a_2	a_1	−	−	1
a'_1	1	−	−	−	−	−
a'_2	−	1	−	−	−	−
a'_3	−	−	1	−	−	−

where $a_i = (i, 0)$ and $a'_i = (i, 1)$ for $i = 1, 2, 3$. In effect algebras $E(Q, \cdot_1)$ and $E(Q, \cdot_2)$ partial order $\leq$ is the same: a_1, a_2, a_3 are minimal nonzero elements, a'_1, a'_2, a'_3 are maximal elements not equal to 1, moreover $a_i \leq a'_j$ for all $i, j \in \{1, 2, 3\}$. Obviously orthosupplementation $'$ is the same in both effect algebras mentioned above. But $E(Q, \cdot_1)$ and $E(Q, \cdot_2)$ are not isomorphic:

Suppose that a mapping $\phi\colon (Q\times\{0\})\cup(Q\times\{1\})\cup\{0\}\cup\{1\}\to (Q\times\{0\})\cup(Q\times\{1\})\cup\{0\}\cup\{1\}$ is an isomorphism of $E(Q,\cdot_1)$ onto $E(Q,\cdot_2)$. Then

$$\phi(a_1)\oplus_2\phi(a_1)=\phi(a_1\oplus_1 a_1)=\phi(a_1)=\phi(a_1)\oplus_2 0$$

so $\phi(a_1)=0$, but $\phi(0)=0$ hence $a_1=0$ and we obtain a contradiction.

So in fact effect algebras $E(Q,\cdot_1)$ and $E(Q,\cdot_2)$ are not isomorphic and it follows that in some effect algebras partial order $\leq$, and orthosupplementation $'$ do not determine $\oplus$.

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